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A 3D ISODIAMETRIC PROBLEM WITH ADDITIONAL CONSTRAINTS

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Abstract. For $\theta \in (\frac{\pi}{3}, \pi)$ let C_θ be the infinite right circular cone in $\mathbb{R}^3$ with aperture θ, which is symmetric about the x-axis and has its apex at the origin O. Among the compact convex domains Δ of diameter 1 with $O \in \Delta \subset C_\theta$, we show that there is a unique domain Δ_θ with maximal volume. Moreover, we give complete information on the shape of Δ_θ and compute the volume.

1. Introduction and Main Result

The *classical isodiametric problem in* $\mathbb{R}^n$ asks for compact domains of diameter one that have maximum n-dimensional volume. It is well known that the unique solution is a closed ball of diameter one.

Among compact convex domains Δ of diameter 1 with $(0,0) \in \Delta \subset \left\{(u, u\tan s) \in \mathbb{R}^2 : \ u \geq 0, \quad |s| \leq \frac{\theta}{2}\right\}$, Dai-He-Luo [2] determine the unique one with maximum area, say Σ_θ, give detailed information on how to construct Σ_θ, and explicitly represent the area of Σ_θ as a function of $\theta \in (0, \pi)$. In particular, it is easy to check that $\Sigma_\theta = \left\{(u\cos s, u\sin s) \in \mathbb{R}^2 : \ 0 \leq u \leq 1, \quad |s| \leq \frac{\theta}{2}\right\}$ for $\theta \leq \frac{\pi}{3}$.

Going to dimension three, we may ask for compact convex domains of diameter 1 with $O \in \Delta \subset C_\theta$ that have maximum volume, where

$$C_\theta = \left\{(u, u\tan s\sin t, u\tan s\cos t) \in \mathbb{R}^3 : \ u \geq 0, \quad |s| \leq \frac{\theta}{2}, \quad 0 \leq t \leq 2\pi\right\}.$$

Let $\mathcal{M}_\theta$ be the collection of all those Δ.

Considering $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$ we can present the main result as follows.

Theorem 1.1. *For $\theta \in (0, \pi)$ the solid Δ_θ formed by rotating Σ_θ about the x-axis is the only element in $\mathcal{M}_\theta$ having maximum volume.*

The cases when $\theta \leq \frac{\pi}{3}$ are trivial so we assume that $\frac{\pi}{3} < \theta < \pi$ hereafter. And it is not difficult to see that similar discussions also work for higher dimensional versions of Theorem 1.1. Besides those, we want to make two further notes.

Firstly, $X = \{(x_1, x_2, x_3) \in C_\theta : x_1^2 + x_2^2 + x_3 \leq 1\}$ is a compact set in $\mathbb{R}^3$ and the family $\mathcal{H}$ of all its nonempty compact subsets forms a compact space under Hausdorff distance d_H[4, p.281,Exercise 7(d)]. Then $\mathcal{M}_\theta$ is closed and hence compact in the metric space $(\mathcal{H}, d_H)$. Since $\Delta \mapsto \text{Area}(\Delta)$ defines an upper semi-continuous function, there is an element in $\mathcal{M}_\theta$ with maximum volume.

Secondly, the proof for Theorem 1.1 crucially depends on random Steiner symmetrization [3]. This makes the major difference of our study from that by Dai-He-Luo [2].

Our paper is organized as follows. Section 2 discusses a generalization of Steiner symmetrization in $\mathbb{R}^3$ and obtains a property that an elements of $\mathcal{M}_\theta$ with maximal volume necessarily has. Section 3 gives a complete proof for Theorem 1.1.

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2. Steiner Symmetrization and Information on "Good Shapes"

We recall two different approaches to find *symmetrized analogs* of a convex body C in $\mathbb{R}^n$, according to a line L through the origin and the hyperplane Π (with codimension one) that goes through the origin and is perpendicular to L.

The first approach considers each of the lines L_x that is parallel to L and goes through a point x on the hyperplane Π. If $L_x \cap C$ is nonempty, let A_x be the line segment with middle point x whose length is equal to that of $L_x \cap C$. The union of those segments A_x is called the **Steiner symmetrization of C about L**, denoted as $\mathcal{S}_L(C)$.

The second approach considers each of the hyperplanes Π_x that is parallel to Π and goes through a point $x \in L$. If $\Pi_x \cap C$ is nonempty, let $B_x \subset \Pi$ be an $(n-1)$-dimensional ball with volume equal to the $(n-1)$-dimensional volume of $\Pi_x \cap C$; otherwise, let B_x be the empty set. Then the union $\bigcup_x (x + B_x)$ is called the **Steiner symmetral of C about L**, denoted as $\mathcal{S}_L^*(C)$.

It is well known that the diameter of $\mathcal{S}_L(C)$, denoted $|S_L(C)|$, is smaller than or equal to $|C|$ while the volume of $\mathcal{S}_L(C)$ is equal to that of C. Clearly, the volume of $\mathcal{S}_L^*(C)$ is equal to that of C. We may focus on the case $n = 3$ and obtain verify the inequality $|\mathcal{S}_L^*(C)| \leq |C|$ in a few steps, by routine checkings.

Step-1. The inequality $d^*(\mathcal{S}_L(X), \mathcal{S}_L(Y)) \leq d^*(X, Y)$ holds for compact convex sets $X, Y \subset \mathbb{R}^n$, where $d^*(X, Y) = \sup\{\|x - y\| : x \in X, y \in Y\}$.

Step-2. According to Peter Mani-Levitska [3], sequences of random Steiner symmetrizations of a convex domain $K \subset \mathbb{R}^2$ converge almost surely in the Hausdorff distance to the disk B_K, whose area is equal to that of K, assuming that the directions are chosen independently and uniformly.

Step-3. For compact convex sets $X, Y \subset \mathbb{R}^2$ let B_X and B_Y be the disks centered at the origin O whose areas are respectively equal to the area of X and that of Y. Then step-2 leads to $d^*(B_X, B_Y) \leq d^*(X, Y)$, which further leads to $|\mathcal{S}_L^*(C)| \leq |C|$.

In the rest part of this section, we always assume that Δ is an element in $\mathcal{M}_\theta$ having maximum volume. We will obtain basic and important properties of its Steiner symmetral Δ^* about the x-axis.

The following lemmas can be directly verified. See [2] for two dimensional analogs.

Lemma 2.1. *There is a point $p \in \Delta^*$ that lies on the unit sphere $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.*

Lemma 2.2. *Let $a \in [0, 1]$ be the largest number such that $\{(x_1, x_2, x_3) \in C_\theta : x_1 \leq a\}$ is contained in Δ^*. Then for each point p in $\{(x_1, x_2, x_3) \in \partial\Delta^* : x_1 \geq a\}$ there is a point $q \in \partial\Delta^*$ such that the segment $\overline{pq}$ between p and q is of length $|pq| = 1$.*

Lemma 2.3. *Let $b \in [0, 1]$ be the smallest number such that $\Delta^* \cap \{(x_1, x_2, x_3) : x_1 = b, x_1^2 + x_2^2 + x_3^2 = 1\} \neq \emptyset$. Then Δ^* contains $\{(x_1, x_2, x_3) \in A_\theta : x_1 \geq b\}$ as a subset, where $A_\theta := \{(x_1, x_2, x_3) \in C_\theta : x_1^2 + x_2^2 + x_3^2 \leq 1\}$. In particular, the point $(1, 0, 0)$ belongs to $\partial\Delta^*$.*

Let a be defined as in Lemma 2.2, and b defined as in Lemma 2.3. Clearly, the two planes $\Pi_a = \{(x_1, x_2, x_3) : x_1 = a\}$ and $\Pi_b = \{(x_1, x_2, x_3) : x_1 = b\}$ each intersect Δ^* at a disk. Denote those two disks by D_a and D_b, respectively.

Lemma 2.4. *The diameter of D_a is equal to that of D_b.*

Proof. Firstly, we show that $|D_a| \leq |D_b|$. If on the contrary $|D_a| > |D_b|$ we may choose $\epsilon_1 > 0$ with $|D_a| > |D_b| + 3\epsilon_1$ such that
(1) the plane $\Pi_1 = \{(x_1, x_2, x_3) : x_1 = b - \varepsilon_1\}$ intersects Δ^* at a disk with diameter less than $|D_b| + \varepsilon_1$.
(2) the plane $\Pi_2 = \{(x_1, x_2, x_3) : x_1 = a - \varepsilon_1\}$ intersects Δ^* at a disk with diameter larger than $|D_a| - \varepsilon_1$.

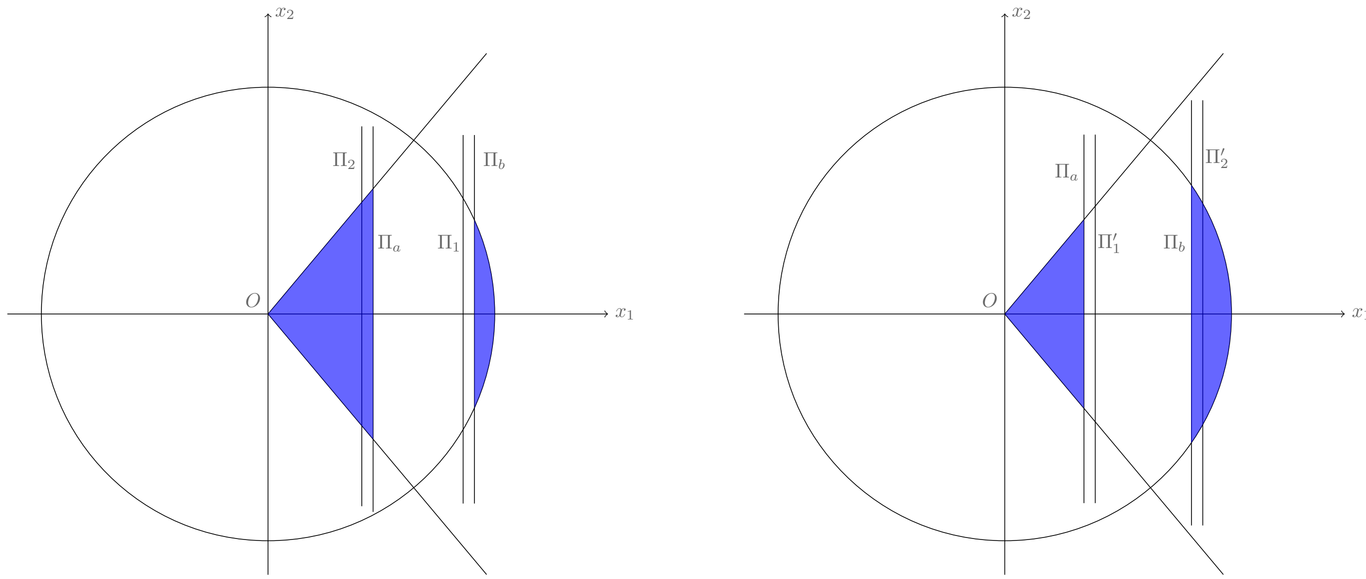

FIGURE 1. How to choose the planes Π_1, Π_2, Π_1' and Π_2'.

See left part of Figure 1 for the intersection of the Ox_1x_2 plane with the planes Π_1, Π_2, Π_a and Π_b.

Since $\max\{|pq| : p \in \Pi_1 \cap \Delta^*, q \in \Delta^*\} < 1 - \varepsilon_2$ for some number $\epsilon_2 > 0$, we may put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and verify that $X = \Delta_l \cup \Delta_c \cup \Delta_r$ is of diameter one, where

$$\Delta_l = \{(x_1, x_2, x_3) \in \Delta^* : x_1 \leq a - \varepsilon_1\};$$
$$\Delta_r = \{(x_1, x_2, x_3) \in \Delta^* : x_1 \geq b - \varepsilon_1 + \varepsilon\};$$
$$\Delta_c = \{(x_1 + \varepsilon, x_2, x_3) : (x_1, x_2, x_3) \in \Delta^*, x_1 \leq b - \varepsilon_1\}.$$

By the choice of ε_1, we may check that the convex set spanned by X would have a volume greater than that of Δ^*. This is impossible.

Then, we continue to show that $|D_a| \geq |D_b|$. If on the contrary $|D_a| < |D_b|$ we may choose $\epsilon_1 > 0$ with $|D_a| < |D_b| - 3\epsilon_1$ such that

- the plane $\Pi_1' = \{(x_1, x_2, x_3) : x_1 = a + \varepsilon_1\}$ intersects Δ^* at a disk with diameter less than $|D_a| + \varepsilon_1$.
- the plane $\Pi_2' = \{(x_1, x_2, x_3) : x_1 = b + \varepsilon_1\}$ intersects Δ^* at a disk with diameter larger than $|D_b| - \varepsilon_1$.

See right part of Figure 1 for the intersection of the Ox_1x_2 plane with the planes Π_1', Π_2', Π_a and Π_b.

We claim that there is a number $\varepsilon_2 > 0$ such that $|pq| \leq 1 - \varepsilon_2$ for any $p \in \Pi_2' \cap \Delta^*$ and any $q = (x_1, x_2, x_3) \in \Delta^*$ with $x_1 \geq a + \varepsilon_1$. Otherwise, there were a point $p \in \Pi_2' \cap \Delta^*$ and a point $q = (x_1, x_2, x_3) \in \Delta^*$ with $x_1 \geq a + \varepsilon_1$ such that $|pq| = 1$. Here we may further check that the segment $\overline{pq}$ intersects the first coordinate axis. Fix a point $p' = (b, x_2, x_3) \in \partial\Delta^*$ which lies in the plane Π^* spanned by $\overline{pq}$ and the first coordinate axis. Then the origin O and the four points p, q, p' and $q' = (b, -x_2, -x_3)$ live on the plane Π^*. However, the segment $\overline{pq}$ would miss either $\overline{Op'}$ or $\overline{Oq'}$, hence the diameter of $\overline{pq} \cup \overline{Op'} \cup \overline{Oq'}$ would be greater than one. This is impossible.

Choose a small enough $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ such that $\Delta_c' = \{(x_1 - \varepsilon, x_2, x_3) : (x_1, x_2, x_3) \in \Delta^*, x_1 \geq a + \varepsilon_1\}$ is contained in C_θ and that $X = \{(x_1, x_2, x_3) \in \Delta^* : x_1 \leq a\} \cup \Delta_c' \cup \{(x_1, x_2, x_3) \in \Delta^* : x_1 \geq b + \varepsilon_1\}$ is of diameter one. By the choice of ε_1, we can see that the convex set spanned by X would have a volume greater than that of Δ^*. This is impossible. □

Lemma 2.5. $|D_a| = |D_b| > 0$.

Proof. Otherwise, $a = 0$ and $b = 1$; hence Δ^* intersects ∂C_θ at a single point, the origin O; and it intersects the unit sphere $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ at a single point $(1, 0, 0)$. Choose a point $p_s = (s, y_s, 0)$ on the boundary $\partial\Delta^*$ with $s \in (0, 1)$, then we have $|y_s| \le s \tan \frac{\theta}{2}$. Here θ is the aperture of the cone C_θ.

Since $a = 0$, we may choose a point q_s on Δ^* with $|p_s q_s| = 1$. As Δ^* is a solid of revolution, the point q_s lies on the Ox_1x_2 plane. We write its coordinate as $(\cos\beta_s, y'_s, 0)$ for some $\beta_s \in (0, \frac{\pi}{2})$. Clearly, $|y'_s| \le \sin\beta_s$.

Now we can see that as $s \to 0$, the point q_s approaches $(1, 0, 0)$. That is to say, $\lim_{s\to 0} \beta_s = 0$. Let $f(s) = |p_s q_s|$. Then $f(s) \equiv 1$. However, for every s we have

$$
\begin{aligned}
|p_s q_s|^2 &= (s - \cos\beta_s)^2 + (y_s - y'_s)^2 = s^2 - 2s\cos\beta_s + \cos^2\beta_s + y_s^2 - 2y_s y'_s + y'^2_s \\
&\le s^2 - 2s\cos\beta_s + \cos^2\beta_s + s^2\tan^2\frac{\theta}{2} + 2s\tan\frac{\theta}{2}\sin\beta_s + \sin^2\beta_s \\
&= 1 + s^2\left(1 + \tan^2\frac{\theta}{2}\right) + 2s\tan\frac{\theta}{2}\sin\beta_s - 2s\cos\beta_s.
\end{aligned}
$$

The right most formula is less than 1 for small enough $s > 0$. This provides a contradiction. □

Lemma 2.6. *Let $p_\pm = (a, \pm a\tan\frac{\theta}{2}, 0)$ be the two points on $D_a \cap \partial C_\theta$. Let $q_\pm = (b, \pm c, 0)$ be the two points on $D_b \cap \partial A_\theta$. Then $|p_+q_-| = |p_-q_+| = 1$ and $a = \dfrac{1+\cos\theta}{\sqrt{5 - 4\cos\theta}}$.*

Proof. By the choice of a in Lemma 2.2, there is a point $q = (x_1, x_2, 0)$ on $\partial\Delta^*$ with $|p_+q| = 1$. Since $\overline{p_+q}$ intersects the first coordinate axis and each of the two segments $\overline{Oq_\pm}$, we have $x_1 \le b$ and $x_2 < 0$. Suppose that $q \ne q_-$. See the following picture for relative locations of five points $p_\pm, q_\pm$ and q on the Ox_1x_2 plane.

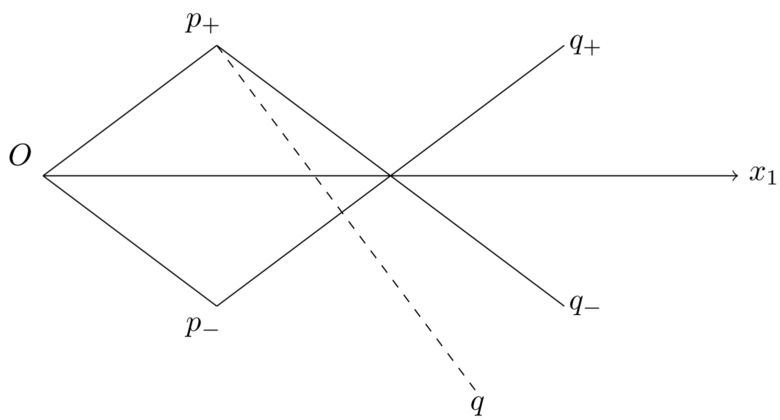

FIGURE 2. Relative locations of $p_\pm, q_\pm$, and q.

Then, for each point $q' = (y_1, y_2, 0) \in \partial\Delta^*$ with $x_1 < y_1 < b$ we choose a point $z = (z_1, z_2, 0) \in \partial\Delta^*$ with $|q'z| = 1$. As the segment $\overline{q'z}$ intersects the first coordinate axis and the segments $\overline{p_+q}$, we have $0 \le z_1 \le a$ and $z_2 > 0$. This indicates that either $z' = O$ or $z' = p_+$. Since $y_2 < b$ we have $|Oq'| < 1$ and hence $z = p_+$.

Finally, by direct calculations we can verify that $a = \sqrt{\dfrac{2+2\cos\theta}{5-4\cos\theta}}\cos\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{\sqrt{5-4\cos\theta}}$. □

3. The shape of Δ and the proof of the main theorem

Let $a = \dfrac{1+\cos\theta}{\sqrt{5-4\cos\theta}}$ and $b = \dfrac{2-\cos\theta}{\sqrt{5-4\cos\theta}}$.

The results from Section 2 indicate that Δ contains the right circular cone $U_l := \{(x_1, x_2, x_3) \in C_\theta : x_1 \le a\}$ and the spherical cap $U_r := \{(x_1, x_2, x_3) : x_1 \ge b,\ x_1^2 + x_2^2 + x_3^2 \le 1\}$. See the following picture for the intersection of the Ox_1x_2 plane with U_l and U_r.

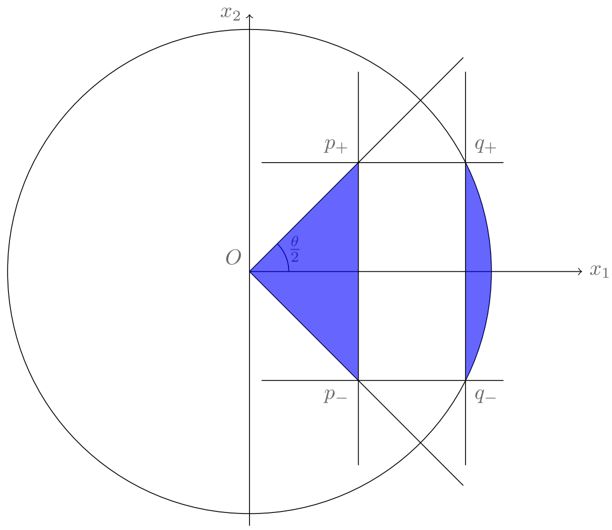

FIGURE 3. A depiction for U_l and U_r on the Ox_1x_2 plane.

The remaining task is to fully understand the shape of $U_c := \{(x_1, x_2, x_3) \in \Delta : a \leq x_1 \leq b\}$.

To this end, we consider the collection $\mathcal{D}$ of all the convex bodies $U \subset W := \{(x_1, x_2, x_3) : a \leq x_1 \leq b\}$ with diameter $|U| = 1$ such that $U \cap \Pi_a = U_l \cap \Pi_a$ and $U \cap \Pi_b = U_r \cap \Pi_b$. Here Π_a is the plane $\{x_1 = a\}$ and Π_b is the plane $\{x_1 = b\}$.

Theorem 3.1. *Let B be the solid ball centered at $(\frac{a+b}{2}, 0, 0)$ with radius $\frac{1}{2}$. Then $B \cap W$ is the unique element of $\mathcal{D}$ with the maximal volume.*

Proof. Given a convex body $U \in \mathcal{D}$. For each point $p \in U$, we may fix the first coordinate and appropriately choose the second and the third coordinate axes such that p is written as $(y_1, y_2, 0)$ with $y_2 > 0$. By the definition of $\mathcal{D}$, we also have $a \leq y_1 \leq b$.

We claim that the union of U with the two spherical caps $U_a := \{(x_1, x_2, x_3) \in B : x_1 \leq a\}$ and $U_b = \{(x_1, x_2, x_3) \in B : x_1 \geq b\}$ is of diameter one. Here, we only need to verify that $|U \cup U_b| \leq 1$. Actually, for any point $q = (b + \epsilon, z_2, z_3) \in U_b$ we have

$$\begin{aligned}
|pq|^2 &= (y_1 - b - \epsilon)^2 + (y_2 - z_2)^2 + z_3^2 \\
&= (b + \epsilon)^2 + y_1^2 - 2y_1(b + \epsilon) + y_2^2 + z_2^2 - 2y_2z_2 + z_3^2 \\
&= 1 + y_1^2 + y_2^2 - 2y_1(b + \epsilon) - 2y_2z_2.
\end{aligned}$$

Since the point $q_- = (b, -\sqrt{1 - b^2}, 0)$ belongs to $D_b \subset U$, we also have $|qq_-|^2 = (y_1 - b)^2 + (y_2 + \sqrt{1 - b^2})^2 \leq 1$, which indicates that $y_1^2 + y_2^2 \leq 2y_1b - 2y_2\sqrt{1 - b^2}$. Combining this with the above formula, we have $|pq| \leq 1$.

Therefore, $U \cup U_a \cup U_b$ is a compact convex body of diameter one. By Biebebach's solution to the classical isodiametric problem [1], we can infer that $U = B \cap W$ is the only element of $\mathcal{D}$ with maximal volume. $\square$

As $\Delta \cap \{x_1 \leq a\} = U_l$ and $\Delta \cap \{x_1 \geq b\} = U_r$, we may present the proof for Theorem 1.1 in two lines.

Proof. Theorem 3.1 indicates that $B \cap W$ has a volume strictly larger than that of any other convex body $U \in \mathcal{D}$. Since Δ has the maximal volume, we have $\Delta \cap W = B \cap W$ and hence $\Delta = U_l \cup (B \cap W) \cup U_r = \Delta_\theta$. $\square$

4. APPENDIX

Intuitively, the convex domain Δ_θ may be obtained by the following steps:

Step 1. Draw the unit ball B_1 and a ball $B_\star$ of radius $\frac{1}{2}$ centered at $(2,0,0)$.

Step 2. Shift the center of $B_\star$ along the first coordinate axis toward the origin until the sphere ∂B_* is tangent to the slant surface of C_θ. By now $\partial B_\star$ intersects ∂B_1 at a circle C_1 of radius $r_1 > 0$, and intersects ∂C_θ at a circle C_2 of radius $r_2 > 0$. Here, one may verify that $r_1 > r_2 > 0$. See left part of Figure 4 for the sections of the balls on the Ox_1x_2 plane.

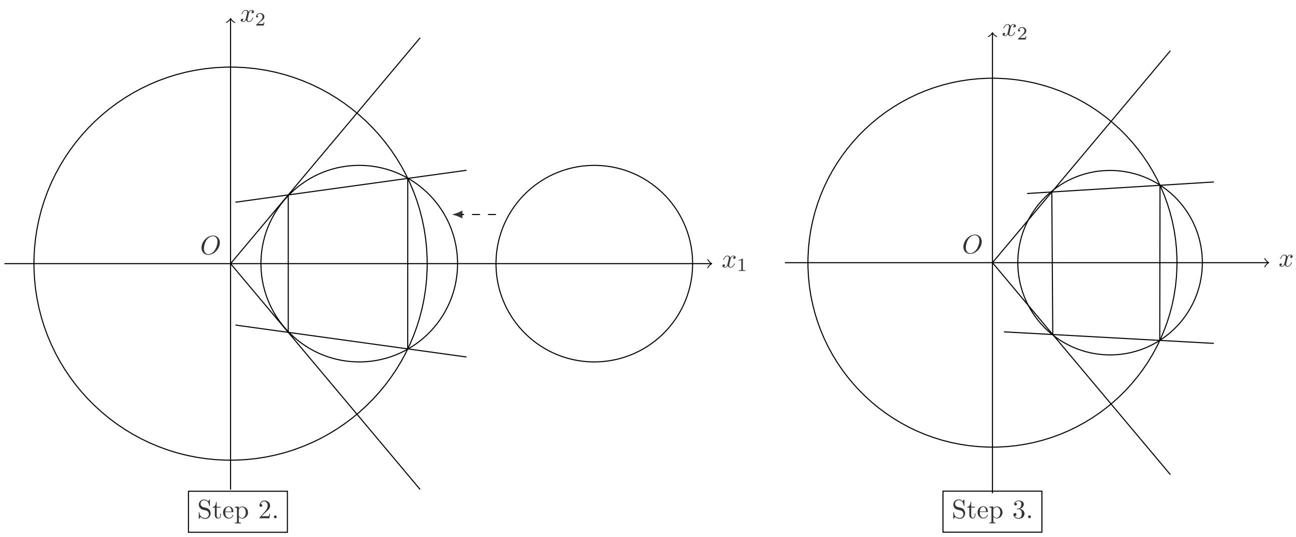

FIGURE 4. Hints on how the ball $B_\star$ moves.

Step 3. Shift $B_\star$ slightly further. The sphere $\partial B_\star$ now intersects ∂B_1 at a circle C_1 with radius $r_1 > 0$ and intersects ∂C_θ at two circles, the larger one C_2 has a radius $r_2 > 0$. Here, we still have $r_1 > r_2 > 0$. See right part of Figure 4.

Step 4. Continuing to shift $B_\star$ until its center is located at q, we have $r_1 = r_2 = \dfrac{\sin\theta}{\sqrt{5-4\cos\theta}}$. Moreover, the circle C_1 and C_2 lie in the planes Π_1 and Π_2 respectively. See Figure 5 for the intersection of Δ_θ with the Ox_1x_2 plane, in which the point q is marked by a dot.

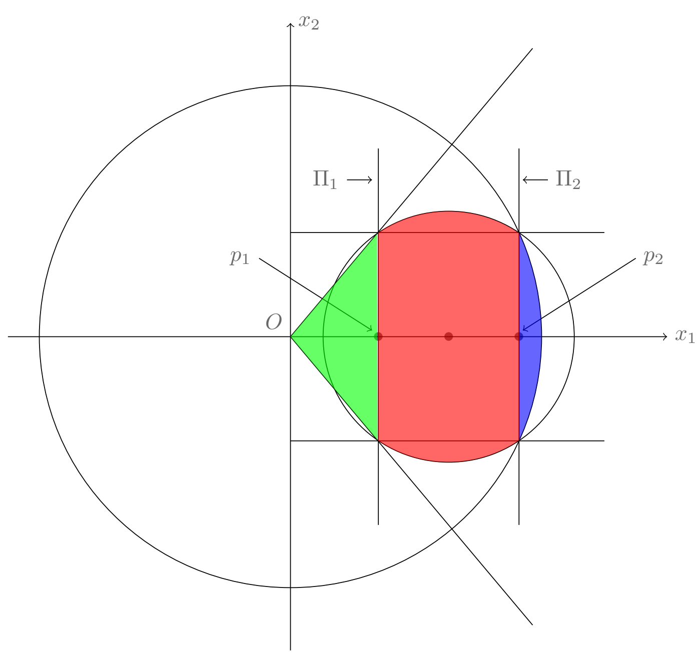

FIGURE 5. The section of Δ_θ on the Ox_1x_2 plane.

Respectively call the **green**, **red** and **blue** domains **(part 1)**, **(part 2)** and **(part 3)**.

By direct calculations we may respectively obtain the volumes of **(part 1)**, **(part 2)** and **(part 3)**:

$$\frac{\pi(1+\cos\theta)\sin^2\theta}{3\times(5-4\cos\theta)^{3/2}}, \quad \frac{\pi(7-18\cos\theta+6\cos^2\theta+4\cos^3\theta)}{6\times(5-4\cos\theta)^{3/2}}, \quad \frac{2\pi}{3}+\frac{\pi(2-\cos\theta)^3}{3\times(5-4\cos\theta)^{3/2}}-\frac{\pi(2-\cos\theta)}{\sqrt{5-4\cos\theta}}.$$

This means that the volume of Δ_θ is exactly $\frac{2\pi}{3}-\frac{\pi(35-38\cos\theta+8\cos^2\theta)}{6\times(5-4\cos\theta)^{3/2}}$. ♯

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